Meeting 12 Math 22

Reading Debrief

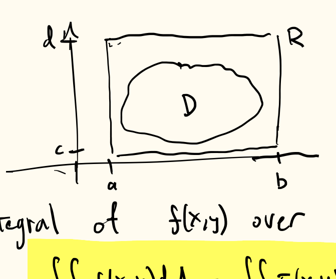
- Discuss Activity 11.2.2 and Activity 11.2.3 w/ your group
- Questions from Section 11.2?

Section 11.3.1 Double Integrals over General Regions

Let D be a closed and bounded region and $f(x,y)$ which is continuous on D. We define $\iint_D f(x,y)\,dA$ as follows: Define a new function $F(x,y)$,

$$F(x,y) = \begin{cases} f(x,y), & (x,y)\in D \\ 0, & \text{otherwise} \end{cases}$$

Choose a rectangle $R=[a,b]\times[c,d]$ which contains D

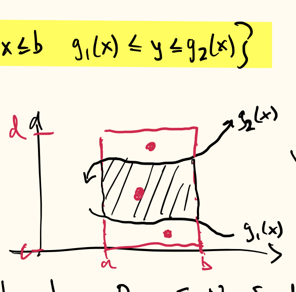


The double integral of $f(x,y)$ over D is

$$\iint_D f(x,y)\,dA = \iint_R F(x,y)\,dA = \int_a^b\int_c^d F(x,y)\,dy\,dx.$$

Type I Region Suppose D is a region which is bounded above and below by two continuous functions $g_1(x)$ and $g_2(x)$ on an interval $[a,b]$

$D=\{(x,y) \mid a\le x\le b \quad g_1(x)\le y\le g_2(x)\}$

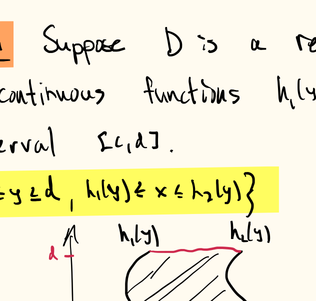


Choose a rectangle $R=[a,b]\times[c,d]$ w/ $c\le g_1(x)\le g_2(x)\le d$. Then

$$\iint_D f(x,y)\,dA = \iint_R F(x,y)\,dA$$
$$= \int_a^b\int_c^d F(x,y)\,dy\,dx$$
$$= \int_a^b\left(\int_c^{g_1(x)} F\,dy + \int_{g_1(x)}^{g_2(x)} F\,dy + \int_{g_2(x)}^d F\,dy\right)dx$$
$$= \int_a^b\left(0 + \int_{g_1(x)}^{g_2(x)} f(x,y)\,dy + 0\right)dx$$
$$= \int_a^b\int_{g_1(x)}^{g_2(x)} f(x,y)\,dy\,dx$$

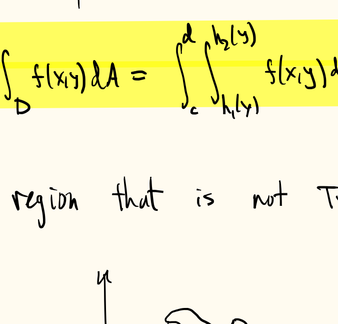
Type II Region Suppose D is a region bounded by two continuous functions $h_1(y)$ and $h_2(y)$ on an interval $[c,d]$.

$D=\{(x,y) \mid c\le y\le d,\ h_1(y)\le x\le h_2(y)\}$



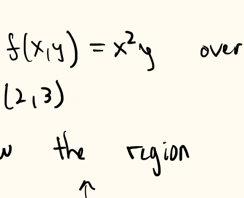
then
$$\iint_D f(x,y)\,dA = \int_c^d\int_{h_1(y)}^{h_2(y)} f(x,y)\,dx\,dy$$

Example A region that is not Type I or Type II



Example $f(x,y)=x^2y$ over a triangle $(0,0),(2,0),(2,3)$

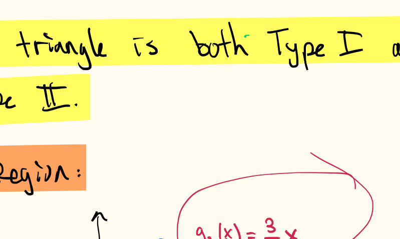
Step 1: Draw the region



Step 2: Determine if the region is Type I or Type II.

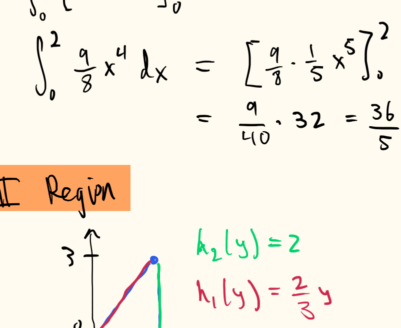
The triangle is both Type I and Type II.

As Type I Region:



$$\iint_T x^2y\,dA = \int_0^2\int_0^{\frac{3}{2}x} x^2y\,dy\,dx$$
$$= \int_0^2\left[\frac{1}{2}x^2y^2\right]_0^{3/2x}dx$$
$$= \int_0^2 \frac{9}{8}x^4\,dx = \left[\frac{9}{8}\cdot\frac{1}{5}x^5\right]_0^2$$
$$= \frac{9}{40}\cdot 32 = \frac{36}{5}$$

As a Type II Region



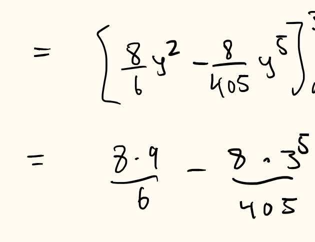
$$\iint_T x^2y\,dA = \int_0^3\int_{\frac{2}{3}y}^{2} x^2y\,dx\,dy$$
$$= \int_0^3\left[\frac{1}{3}x^3y\right]_{\frac{2}{3}y}^{2}dy$$
$$= \int_0^3\left[\frac{8}{3}y - \frac{8}{81}y^4\right]dy$$
$$= \left[\frac{8}{6}y^2 - \frac{8}{405}y^5\right]_0^3$$
$$= \frac{8\cdot 9}{6} - \frac{8\cdot 3^5}{405} = \frac{36}{5}.$$

Activity 11.3.2

- Complete w/ your group.
- Class discussion.

Activity 11.3.2. Consider the double integral $\iint_D (4-x-2y)\,dA$, where D is the triangular region with vertices (0,0), (4,0), and (0,2).

(a) **Type I**



$y=-\frac{1}{2}x+2$
$x=-2(y-2)$
$=-2y+4$

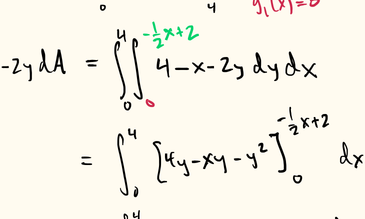
$2\int u^2\,du = \frac{2}{3}u^3$

$$\iint_T 4-x-2y\,dA = \int_0^4\int_0^{-\frac{1}{2}x+2} 4-x-2y\,dy\,dx$$
$$= \int_0^4\left[4y-xy-y^2\right]_0^{-\frac{1}{2}x+2}dx$$
$$= \int_0^4 4\left(-\tfrac{1}{2}x+2\right) - x\left(-\tfrac{1}{2}x+2\right) + \left(-\tfrac{1}{2}x+2\right)^2 dx$$

$u=-\frac{1}{2}x+2$
$du=-\frac{1}{2}dx$

$$= \left[4\left(-\tfrac{1}{4}x^2+2x\right) + \tfrac{1}{6}x^3 - x^2 - \tfrac{2}{3}\left(-\tfrac{1}{2}x+2\right)^3\right]_0^4$$
$$= 4(-4+8) + \tfrac{1}{6}\cdot 64 - 16 - \tfrac{16}{3}$$
$$= 16 + \tfrac{32}{3} - 16 - \tfrac{16}{3} = \tfrac{16}{3}$$

Type II Region

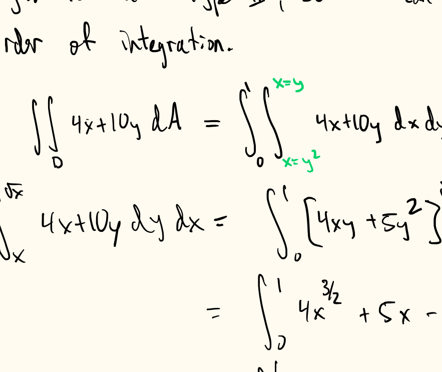


$$\iint_T 4-x-2y\,dA = \int_0^2\int_0^{-2y+4} 4-x-2y\,dx\,dy$$
$$= \int_0^2\left[4x-\tfrac{1}{2}x^2-2xy\right]_0^{-2y+4}dy$$
$$= \int_0^2 -8y+16 - \tfrac{1}{2}(-2y+4)^2 - 2y(-2y+4)\,dy$$
$$= -4y^2+16y + \tfrac{1}{4}\cdot\tfrac{1}{3}(-2y+4)^3 + \tfrac{4}{3}y^3 - 4y^2\Big|_0^2$$
$$= -16+32+0+\tfrac{32}{3}-16-\tfrac{64}{12}$$
$$= \tfrac{16}{3}.$$

Activity 11.3.3

- Complete w/ your group.
- Class discussion.

(a)



Type I

(b) The region is also Type II, so we can swap the order of integration.

$$\iint_D 4x+10y\,dA = \int_0^1\int_{x=y^2}^{x=y} 4x+10y\,dx\,dy$$

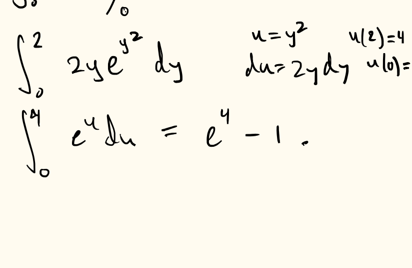
(c)
$$\int_0^1\int_x^{\sqrt{x}} 4x+10y\,dy\,dx = \int_0^1\left[4xy+5y^2\right]_x^{\sqrt{x}}dx$$
$$= \int_0^1 4x^{3/2}+5x-4x^2-5x^2\,dx$$
$$= \int_0^1 4x^{3/2}+5x-9x^2\,dx$$
$$= \tfrac{8}{5}x^{5/2}+\tfrac{5}{2}x^2-3x^3\Big|_0^1$$
$$= \tfrac{8}{5}+\tfrac{5}{2}-3 = \tfrac{11}{10}$$

$$\int_0^1\int_{y^2}^{y} 4x+10y\,dx\,dy = \int_0^1\left[2x^2+10xy\right]_{y^2}^{y}dy$$
$$= \int_0^1 2y^2+10y^2-2y^4-10y^3\,dy$$
$$= \tfrac{2}{3}y^3+\tfrac{10}{3}y^3-\tfrac{2}{5}y^5-\tfrac{10}{4}y^4\Big|_0^1$$
$$= \tfrac{2}{3}+\tfrac{10}{3}-\tfrac{2}{5}-\tfrac{10}{4}$$
$$= 4-\tfrac{58}{20} = \tfrac{22}{20} = \tfrac{11}{10}.$$

Activity 11.3.4

- Complete w/ your group.
- Class discussion.

(b)



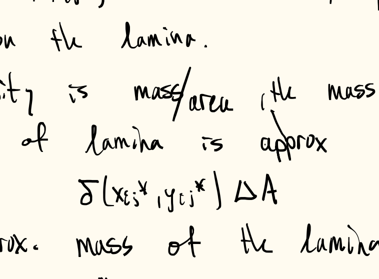
(c)
$$\iint_D e^{y^2}\,dA = \int_0^2\int_0^{2y} e^{y^2}\,dx\,dy$$
$$= \int_0^2 xe^{y^2}\Big|_0^{2y}dy$$
$$= \int_0^2 2ye^{y^2}\,dy$$

$u=y^2$ $u(2)=4$
$du=2y\,dy$ $u(0)=0$

$$= \int_0^4 e^u\,du = e^4-1.$$

Section 11.4.1 Mass

A lamina is a flat thin plate. We can model a lamina with a closed and bounded region D



Suppose $\delta(x,y)$ is a function which gives the density of D at (x,y). Cover D w/ a rectangle R and divide it into $m\cdot n$ subrectangles w/ equal area ΔA. Choose sample points (x_{ij}^*,y_{ij}^*) in each R_{ij}. Define $\delta(x_{ij}^*,y_{ij}^*)=0$ if the sample point does not lie on the lamina.

Since density is mass/area, the mass of each small piece of lamina is approx

$$\delta(x_{ij}^*,y_{ij}^*)\,\Delta A$$

So the approx. mass of the lamina is

$$\sum_{j=1}^{n}\sum_{i=1}^{m}\delta(x_{ij}^*,y_{ij}^*)\,\Delta A$$

Since this is a Riemann sum, we can take a limit to get the exact value of the mass of the lamina

$$\iint_D \delta(x,y)\,dA.$$

Activity 11.4.2

- Complete w/ your group.
- Class discussion.

The mass of the lamina is

$x^2+y^2=9$
$x=\sqrt{9-y^2}$

$$\iint_D x\,dA = \int_{-3}^{3}\int_0^{\sqrt{9-y^2}} x\,dx\,dy$$
$$= \tfrac{1}{2}\int_{-3}^{3} x^2\Big|_0^{\sqrt{9-y^2}}dy$$
$$= \tfrac{1}{2}\int_{-3}^{3} 9-y^2\,dy$$

even function
$$= \int_0^3 9-y^2\,dy = \left[9y-\tfrac{1}{3}y^3\right]_0^3$$
$$= 27-9 = 18$$